Geometric evolution of the Reynolds stress tensor

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Abstract

The dynamics of the Reynolds stress tensor for turbulent flows is described with an evolution equation coupling both geometric effects and turbulent source terms. The effects of the mean flow geometry are shown up when the source terms are neglected: the Reynolds stress tensor is then expressed as the sum of three tensor products of vector fields which are governed by a distorted gyroscopic equation. Along the mean flow trajectories, the fluctuations of velocity are described by differential equations whose coefficients depend only on the mean flow deformation. If the mean flow vorticity is small enough, an approximate turbulence model is derived, and its application to shear shallow water flows is proposed. Moreover, the approximate turbulence model admits a variational formulation which is similar to the one of capillary fluids.

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1 Introduction

The system describing turbulent compressible barotropic flows is composed of equation of mass balance, equation of average momentum and evolution equation for the Reynolds stress tensor. In the following, we see that the Reynolds stress tensor equation is mainly driven by the velocity gradient tensor of the mean motion and this equation is the main object of our study when the source term is negligible.

The reason for considering the simplified turbulence model without the source term is twofold. First, in numerical studies of compressible turbulent

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flows, this homogeneous equation is a natural step in applying the splitting-up technique (see for example [2]). Secondly, such a homogeneous system appears as an exact asymptotic model of weakly shearing flows of long waves over a flat bottom (see [18]). The only difference is the space dimension: two dimensions are considered for shallow water flows instead of three dimensions for the general case.

We use the spectral decomposition of the Reynolds stress tensor and in the homogeneous case, we obtain a simpler dynamical system for the eigenvalues and the eigenvectors of the Reynolds stress tensor. The system admits a simple physical interpretation: the motion of each point of the turbulent flow is analogous to the motion of a free rigid body moving along the mean flow and rotating with an angular velocity which is different from the mean flow vorticity. The angular velocity is completely determined by the mean flow velocity. The moments of inertia of the free rigid body are not constant, they are also determined by the mean flow.

When the mean flow vorticity is small enough, an approximate turbulence model is obtained, which admits a variational formulation.

2 The governing equations

The governing equations of barotropic turbulent compressible fluids are (see [14, 15, 21]) :

$$\left\{\begin{array}{c} \langle\rho\rangle_t + (\langle\rho\rangle\ U_i)_{,i} = 0, \\ \\ (\langle\rho\rangle\, U_i)_t + (\langle\rho\rangle\ U_iU_j + \langle p\rangle\ \delta_{ij} + \langle\rho\, u_iu_j\rangle)\ ,_j = 0, \\ \\ \langle\rho\, u_iu_j\rangle_t + (\langle\rho\, u_iu_j\rangle\ U_k)\ ,_k + \langle\rho\, u_ku_j\rangle\ U_{i,k} + \langle\rho\, u_iu_k\rangle\ U_{j,k} = S_{ij} \end{array}\right. \tag{1}$$

where “brackets” mean the averaging, “coma” means the derivation with respect to the Eulerian coordinates $\mathbf{x} = \{x_i\},\ i \in \{1, 2, 3\}$ and index ”t” means the partial derivative with respect to time, ρ is the fluid density, $\mathbf{U} = \{U_i\},\ i \in \{1, 2, 3\}$ is the mass average velocity, p is the pressure, $\mathbf{u} = \{u_i\},\ i \in \{1, 2, 3\}$ is the velocity fluctuation verifying $\langle\rho\mathbf{u}\rangle = 0$. Repeated indices mean summation. Here $\mathbf{S} = \{S_{ij}\}$ is a source term, and its explicit expression can be written as :

$$S_{ij} = -\,\langle\, u_ip_{,j}\,\rangle - \langle\, u_jp_{,i}\,\rangle - \langle\rho\, u_iu_ju_k\rangle\ ,_k .$$

We introduce the Reynolds stress tensor

$$\mathbf{R} = \langle\rho\ \mathbf{u}\otimes\mathbf{u}\rangle\,, \quad (R_{ij} = \langle\rho\, u_iu_j\rangle).$$

System (1) can be rewritten in the tensorial form

$$\left\{\begin{array}{c}\dfrac{\partial\langle\rho\rangle}{\partial t}+\operatorname{div}\left(\langle\rho\rangle\mathbf{U}\right)=0,\\[2ex] \langle\rho\rangle\dfrac{d\mathbf{U}}{dt}+\nabla\langle p\rangle+\left(\operatorname{div}\mathbf{R}\right)^{T}=\mathbf{0},\\[2ex] \dfrac{d\mathbf{R}}{dt}+\mathbf{R}\operatorname{div}\mathbf{U}+\dfrac{\partial\mathbf{U}}{\partial\mathbf{x}}\mathbf{R}+\mathbf{R}\left(\dfrac{\partial\mathbf{U}}{\partial\mathbf{x}}\right)^{T}=\mathbf{S},\end{array}\right. \tag{2}$$

where d/dt means the material derivative with respect to the mean motion

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{U}^{T}\,\nabla\ .$$

The superscript " T " means the transposition. Using the mass conservation law, the equation for the volume Reynolds stress tensor $\mathbf{R}$ can be rewritten as the equation for the specific (or per unit mass) Reynolds stress tensor

$$\frac{d\mathbf{P}}{dt}+\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\mathbf{P}+\mathbf{P}\left(\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\right)^{T}=\frac{\mathbf{S}}{\langle\rho\rangle}, \tag{3}$$

where

$$\mathbf{P}=\frac{\mathbf{R}}{\langle\rho\rangle}.$$

The structure of source term $\mathbf{S}$ has generated much debate within physical and mathematical communities. Our goal is not to add new closure hypotheses, but to study the structure of the "master" equation (3) when $\mathbf{S}=\mathbf{0}$. The reason is twofold:

- firstly, in the numerical study of compressible turbulent flows, this is a natural step in applying the splitting-up technique (see for example [2]),
- secondly, system (2) also appears as an exact asymptotic model of weakly shearing flows of long waves (turbulent shallow water flows) over a flat bottom (see [18]) :

$$\left\{\begin{array}{c}\dfrac{\partial h}{\partial t}+\operatorname{div}\left(h\,\mathbf{U}\right)=0,\\[2ex] h\dfrac{d\mathbf{U}}{dt}+\nabla\left(\dfrac{gh^{2}}{2}\right)+\left(\operatorname{div}\mathbf{R}\right)^{T}=\mathbf{0},\\[2ex] \dfrac{d\mathbf{R}}{dt}+\mathbf{R}\operatorname{div}\mathbf{U}+\dfrac{\partial\mathbf{U}}{\partial\mathbf{x}}\mathbf{R}+\mathbf{R}\left(\dfrac{\partial\mathbf{U}}{\partial\mathbf{x}}\right)^{T}=\mathbf{0}.\end{array}\right. \tag{4}$$

In system (4), h is the fluid depth playing the role of the average density; the average pressure is given by $\langle p\rangle=gh^{2}/2$, g is the gravity acceleration, and

$$\mathbf{R}=\int_{0}^{h}\left[(\widetilde{\mathbf{U}}-\mathbf{U})\otimes(\widetilde{\mathbf{U}}-\mathbf{U})\right]dz,\qquad h\mathbf{U}=\int_{0}^{h}\widetilde{\mathbf{U}}\,dz,$$

where $\widetilde{\mathbf{U}}$ is the instantaneous velocity. Equations are written for three-dimensional long waves and the production term is zero in the limit of weakly shearing flows. Equations (4) are hyperbolic (see [18] for proof). Finally, we will focus on the equation of the Reynolds stress tensor per unit mass

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{P} + \mathbf{P}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = 0. \tag{5}$$

The case $\mathbf{S} = \mathbf{0}$ corresponds to *conservative motions of turbulent compressible flows*; these motions verify the equations

$$\left\{\begin{aligned} &\frac{\partial \langle\rho\rangle}{\partial t} + \operatorname{div}\left(\langle\rho\rangle\,\mathbf{U}\right) = 0, \\ &\langle\rho\rangle\,\frac{d\mathbf{U}}{dt} + \nabla\langle p\rangle + \left(\operatorname{div}\mathbf{R}\right)^T = \mathbf{0}, \\ &\frac{d\mathbf{R}}{dt} + \mathbf{R}\operatorname{div}\mathbf{U} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\mathbf{R} + \mathbf{R}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = \mathbf{0}. \end{aligned}\right. \tag{6}$$

The particular case $\operatorname{rot}\mathbf{U} = 0$ was investigated in [8]. In such a case $\left(\partial\mathbf{U}/\partial\mathbf{x}\right)^T = \partial\mathbf{U}/\partial\mathbf{x}$ and Eq. (5) corresponds to a two-covariant tensor convected by the mean flow. This means that $\mathbf{P}$ has a zero Lie derivative d_L with respect to the velocity field $\mathbf{U}$ and the tensor $\mathbf{P}_0$, image of $\mathbf{P}$ in Lagrange coordinates $(t, \mathbf{X})$, only depends on $\mathbf{X} = \{X_i\},\ i \in \{1, 2, 3\}$

$$d_L\mathbf{P} \equiv \frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{P} + \mathbf{P}\,\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = 0, \qquad \mathbf{P} = \left(F^T\right)^{-1}\mathbf{P}_0\left(\mathbf{X}\right)F^{-1}, \tag{7}$$

where $F = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient of the mean motion.
The aim of the paper is the study of the homogeneous Reynolds stress tensor equation structure (5) in ***the case*** $\mathtt{rot}\,\mathbf{U} \neq 0$.

3 Geometric properties of the Reynolds stress tensor evolution

The Reynolds stress tensor $\mathbf{P}$ is symmetric and semi-positive definite. The tensor $\mathbf{P}$ can be rewritten in a local basis of orthonormal eigenvectors in the form

$$\mathbf{P} = \sum_{\alpha=1}^{3} \lambda_\alpha^2\ \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha \equiv \sum_{\alpha=1}^{3} \lambda_\alpha^2\ \mathbf{e}_\alpha\,\mathbf{e}_\alpha^T.$$

The eigenvalues λ_α^2, $\alpha \in \{1, 2, 3\}$ are non-negative; the case $\lambda_\alpha^2 > 0$ is a generic one. For the two-dimensional case, $\lambda_3^2 \equiv 0$. Let us denote

$$\mathbf{a}_\alpha = \lambda_\alpha\ \mathbf{e}_\alpha, \quad (\lambda_\alpha > 0) \qquad \alpha \in \{1, 2, 3\}.$$

Then,

$$\mathbf{P} = \sum_{\alpha=1}^{3} \mathbf{a}_\alpha \otimes \mathbf{a}_\alpha \equiv \sum_{\alpha=1}^{3} \mathbf{a}_\alpha\, \mathbf{a}_\alpha^T. \tag{8}$$

From Eq. (8), we deduce

$$\frac{d\mathbf{P}}{dt} = \sum_{\alpha=1}^{3} \left[\frac{d\mathbf{a}_\alpha}{dt}\, \mathbf{a}_\alpha^T + \mathbf{a}_\alpha \left(\frac{d\mathbf{a}_\alpha}{dt} \right)^T \right]. \tag{9}$$

By using Eq. (9), Eq. (5) can be written

$$\sum_{\alpha=1}^{3} \left(\frac{d\mathbf{a}_\alpha}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\, \mathbf{a}_\alpha \right) \mathbf{a}_\alpha^T + \left[\left(\frac{d\mathbf{a}_\alpha}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\, \mathbf{a}_\alpha \right) \mathbf{a}_\alpha^T \right]^T = 0. \tag{10}$$

The vector $d\mathbf{a}_\alpha/dt + (\partial \mathbf{U}/\partial \mathbf{x})\, \mathbf{a}_\alpha$ can be developped in the local basis $\{\mathbf{a}_\beta\}$, $\beta \in \{1, 2, 3\}$ of eigenvectors; one obtains

$$\frac{d\mathbf{a}_\alpha}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\, \mathbf{a}_\alpha = \sum_{\beta=1}^{3} A_{\beta\alpha}\, \mathbf{a}_\beta, \qquad \alpha \in \{1, 2, 3\} \tag{11}$$

where $A_{\beta\alpha}$, $(\alpha, \beta \in \{1, 2, 3\})$ are the scalar components to be determined. By using Eq. (11), Eq. (10) leads to

$$\sum_{\alpha=1}^{3} A_{\alpha\alpha}\, \mathbf{g}_{\alpha\alpha} + \sum_{\alpha\neq\beta=1}^{3} \left(A_{\alpha\beta} + A_{\beta\alpha} \right)\, \mathbf{g}_{\alpha\beta} = 0,$$

where $\mathbf{g}_{\alpha\alpha} = 2\mathbf{a}_\alpha\, \mathbf{a}_\alpha^T$ and $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\beta\alpha} = \mathbf{a}_\alpha\, \mathbf{a}_\beta^T + \mathbf{a}_\beta\, \mathbf{a}_\alpha^T$, $(\alpha, \beta \in \{1, 2, 3\})$, are six independent symmetric tensors. Consequently,

$$A_{\alpha\alpha} = 0 \quad \text{and} \quad A_{\alpha\beta} + A_{\beta\alpha} = 0, \qquad \alpha, \beta \in \{1, 2, 3\}.$$

Equation (10) is equivalent to

$$\frac{d\mathbf{a}_\alpha}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\, \mathbf{a}_\alpha = \Lambda \circ i\,(\pi)\, \mathbf{e}_\alpha \;\text{ with }\; \pi = A_{32}\, \mathbf{e}_1 + A_{13}\, \mathbf{e}_2 + A_{21}\, \mathbf{e}_3, \quad \alpha \in \{1, 2, 3\}, \tag{12}$$

where a diagonal matrix Λ and an antisymmetric matrix $i\,(\pi)$ are determined in the basis $\mathbf{e}_\beta$, $\beta \in \{1, 2, 3\}$ as

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad i\,(\pi) = \begin{pmatrix} 0 & -A_{21} & A_{13} \\ A_{21} & 0 & -A_{32} \\ -A_{13} & A_{32} & 0 \end{pmatrix}.$$

The vectors $\mathbf{a}_\beta$, $\beta \in \{1, 2, 3\}$ are orthogonal, $\mathbf{a}_\alpha^T\, \mathbf{a}_\beta = 0$, $(\alpha \neq \beta)$. This is equivalent to

$$\mathbf{a}_\alpha^T\, \frac{d\mathbf{a}_\beta}{dt} + \mathbf{a}_\beta^T\, \frac{d\mathbf{a}_\alpha}{dt} = 0. \tag{13}$$

Consequently, Eqs. (12) - (13) yield

$$\forall\, \alpha \neq \beta \in \{1,2,3\},$$
$$\mathbf{a}_\alpha^T \left(\Lambda \circ i\,(\pi)\, \mathbf{e}_\beta - \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\ \mathbf{a}_\beta \right) + \mathbf{a}_\beta^T \left(\Lambda \circ i\,(\pi)\, \mathbf{e}_\alpha - \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\ \mathbf{a}_\alpha \right) = 0.$$

Or

$$\begin{aligned} 2\lambda_\alpha \lambda_\beta \mathbf{e}_\alpha^T\ \mathbf{D}\ \mathbf{e}_\beta &= \mathbf{a}_\alpha^T\ \Lambda \circ i\,(\pi)\, \mathbf{e}_\beta + \mathbf{a}_\beta^T\ \Lambda \circ i\,(\pi)\, \mathbf{e}_\alpha \\ &\equiv \lambda_\alpha^2 \mathbf{e}_\alpha^T\ i\,(\pi)\, \mathbf{e}_\beta + \lambda_\beta^2 \mathbf{e}_\beta^T\ i\,(\pi)\, \mathbf{e}_\alpha \end{aligned} \tag{14}$$

where

$$\mathbf{D} = \frac{1}{2} \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T \right)$$

is the rate of deformation tensor corresponding to the mean flow. We denote the mixed product of three vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ as $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv \mathbf{a}^T (\mathbf{b} \wedge \mathbf{c})$. Hence, Eq. (14) yields

$$\begin{aligned} 2\lambda_\alpha \lambda_\beta\, \mathbf{e}_\alpha^T\ \mathbf{D}\ \mathbf{e}_\beta &= \left(\ \lambda_\alpha^2 - \lambda_\beta^2 \right) \left(\mathbf{e}_\alpha, \pi, \mathbf{e}_\beta \right) \\ &= \left(\ \lambda_\beta^2 - \lambda_\alpha^2 \right) \left(\pi, \mathbf{e}_\alpha, \mathbf{e}_\beta \right) \\ &= \left(\ \lambda_\beta^2 - \lambda_\alpha^2 \right) \pi^T \mathbf{e}_\gamma, \end{aligned}$$

where $\{\alpha, \beta, \gamma\}$ is a cyclic permutation of the triplet $\{1,2,3\}$. Finally, we get

$$\pi^T \mathbf{e}_\gamma = \frac{2\lambda_\alpha \lambda_\beta\ \mathbf{e}_\alpha^T\ \mathbf{D}\ \mathbf{e}_\beta}{\lambda_\beta^2 - \lambda_\alpha^2}. \tag{15}$$

Equation (12) can be written

$$\lambda_\alpha \frac{d\mathbf{e}_\alpha}{dt} + \frac{d\lambda_\alpha}{dt}\, \mathbf{e}_\alpha + \lambda_\alpha \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\, \mathbf{e}_\alpha\ - \Lambda \circ i\,(\pi)\, \mathbf{e}_\alpha = 0, \qquad \alpha \in \{1,2,3\}. \tag{16}$$

Since

$$\mathbf{e}_\alpha^T \frac{d\mathbf{e}_\alpha}{dt} = 0, \qquad \mathbf{e}_\alpha^T \Lambda \circ i\,(\pi)\, \mathbf{e}_\alpha = \lambda_\alpha \mathbf{e}_\alpha^T i\,(\pi)\, \mathbf{e}_\alpha = 0,$$

by multiplying the left side of Eq. (16) with $\mathbf{e}_\alpha^T$, we get

$$\frac{d\lambda_\alpha}{dt} + \ \left(\mathbf{e}_\alpha^T\ \mathbf{D}\ \mathbf{e}_\alpha \right) \lambda_\alpha = 0.$$

By multiplying the left side of Eq. (16) with the projector $\left(\mathbf{I} - \mathbf{e}_\alpha \mathbf{e}_\alpha^T\right)$, we get

$$\lambda_\alpha \frac{d\mathbf{e}_\alpha}{dt} + \lambda_\alpha \left(\mathbf{I} - \mathbf{e}_\alpha \mathbf{e}_\alpha^T \right) \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\, \mathbf{e}_\alpha\ - \Lambda \circ i\,(\pi)\, \mathbf{e}_\alpha = 0.$$

We will prove that there exists a vector $\mathbf{\Pi}$ such that

$$\frac{d\mathbf{e}_\alpha}{dt} = \mathbf{\Pi} \wedge \mathbf{e}_\alpha,$$

Such a vector $\mathbf{\Pi}$ should verify the condition

$$\lambda_\alpha i\left(\mathbf{\Pi}\right)\mathbf{e}_\alpha + \lambda_\alpha\left(\mathbf{I} - \mathbf{e}_\alpha\mathbf{e}_\alpha^T\right)\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha \;\; - \Lambda \circ i\left(\pi\right)\mathbf{e}_\alpha = 0. \tag{17}$$

By multiplying Eq. (17) with $\mathbf{e}_\beta^T$ where $\beta \neq \alpha$, we get

$$\lambda_\alpha \mathbf{\Pi}^T \mathbf{e}_\gamma = \lambda_\beta \pi^T \mathbf{e}_\gamma - \lambda_\alpha \mathbf{e}_\beta^T \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha. \tag{18}$$

By replacing Rel. (15) into Eq. (18) we get

$$\begin{aligned}
\mathbf{\Pi}^T \mathbf{e}_\gamma &= \frac{\lambda_\beta^2 \mathbf{e}_\alpha^T \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} + \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T\right)\mathbf{e}_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} - \mathbf{e}_\beta^T \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha \\
&= \frac{\mathbf{e}_\beta^T \left(\lambda_\beta^2 \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T + \lambda_\alpha^2 \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)\mathbf{e}_\alpha}{\lambda_\beta^2 - \lambda_\alpha^2}.
\end{aligned}$$

Now, we can formulate the result,

Theorem 1. *The Reynolds stress tensor can be written in the form*

$$\mathbf{R} = \langle\rho\rangle \sum_{\alpha=1}^{3} \lambda_\alpha^2\; \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha.$$

The eigenvectors $\mathbf{e}_\alpha$ and the eigenvalues λ_α verify the equations:

$$\begin{cases} \dfrac{d\mathbf{e}_\alpha}{dt} = \mathbf{\Pi} \wedge \mathbf{e}_\alpha, \\[2ex] \dfrac{d\left(\mathrm{Ln}\,\lambda_\alpha^2\right)}{dt} = -2\;\mu_\alpha, \end{cases} \qquad \alpha \in \{1, 2, 3\} \tag{19}$$

where

$$\mu_\alpha = \mathbf{e}_\alpha^T\;\mathbf{D}\;\mathbf{e}_\alpha, \qquad \mathbf{\Pi}^T \mathbf{e}_\gamma = \frac{\mathbf{e}_\beta^T \left(\lambda_\beta^2 \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T + \lambda_\alpha^2 \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)\mathbf{e}_\alpha}{\lambda_\beta^2 - \lambda_\alpha^2}.$$

The triplet $\{\alpha, \beta, \gamma\}$ *corresponds to a cyclic permutation of the triplet* $\{1, 2, 3\}$.

Equation (19_1) is similar to the equations of a rigid body (see [13]). The vectors $\mathbf{e}_\alpha$ form a natural moving frame $\{\mathbf{e}_\alpha\}_{\alpha=1}^3$ whose evolution is determined by the mean rate of deformation tensor. The eigenvalues λ_α^2 of the Reynolds stress tensor are determined by the evolution equation (19_2). Let us note that, if λ_α are initially positive, they will be positive for any time. Hence, it means that the tensor P will always be positive definite. Due to the mass conservation law (2_1) and Eq. (19_2), we obtain the following quantity conserved along trajectories of mean flow:

$$\frac{d}{dt}\left(\langle\rho\rangle^{-2}\prod_{\alpha=1}^{3}\lambda_\alpha^2\right) = 0.$$

Consequently, system (19) admits an invariant scalar along the trajectories of mean flow. This invariant was earlier obtained in [7, 8] in a different form. Let us introduce the turbulent specific energy

$$e_T = \frac{1}{2}\,\mathtt{tr}\,\mathbf{P} = \frac{1}{2}\sum_{\alpha=1}^{3}\lambda_\alpha^2.$$

In the incompressible (isochoric) case, we have $d\langle\rho\rangle/dt = 0$; the turbulent energy is minimal in the isotropic case when the three eigenvalues λ_α^2 are equal ($\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda^2$). In this case, the orthonormal eigenvectors $\mathbf{e}_\alpha$, $\alpha \in \{1, 2, 3\}$ of the Reynolds stress tensor $\mathbf{P}$ are also the orthonormal eigenvectors of the mean rate of deformation tensor $\mathbf{D}$ and μ_α are the corresponding eigenvalues.

In the compressible isotropic case $e_T = \langle\rho\rangle^{2/3}\kappa$, $\kappa = 3\lambda^2/(2\,\langle\rho\rangle^{2/3})$, and κ is a classical invariant of isotropic compressible turbulence.

In presence of shock waves the quantity $\langle\rho\rangle^{-2}\prod\limits_{\alpha=1}^{3}\lambda_\alpha^2$ is not conserved through shocks; it increases like the classical entropy in compressible fluid dynamics. The estimation of the jump of turbulence entropy in isotropic case was given in [12].

As a consequence, the governing equations (6) admit the energy conservation law

$$\frac{\partial}{\partial t}\left(\langle\rho\rangle\left(\frac{1}{2}|\mathbf{U}|^2 + e_i + e_T\right)\right) + \mathrm{div}\left(\langle\rho\rangle\mathbf{U}\left(\frac{1}{2}|\mathbf{U}|^2 + e_i + e_T\right) + (\langle p\rangle\mathbf{I} + \mathbf{R})\,\mathbf{U}\right) = 0$$

where the internal specific energy e_i is defined by

$$de_i = -\langle p\rangle\, d\left(\frac{1}{\langle\rho\rangle}\right)$$

and the mean pressure $\langle p\rangle$ is supposed to be a given function of $\langle\rho\rangle$. Indeed, using (19_2) we immediately obtain

$$\frac{\partial}{\partial t}\left(\langle\rho\rangle\left(\frac{1}{2}|\mathbf{U}|^2 + e_i + e_T\right)\right) + \mathrm{div}\left(\langle\rho\rangle\mathbf{U}\left(\frac{1}{2}|\mathbf{U}|^2 + e_i + e_T\right) + (\langle p\rangle\mathbf{I} + \mathbf{R})\,\mathbf{U}\right)$$

$$= \langle\rho\rangle \frac{de_T}{dt} + \mathtt{tr}\,(\mathbf{R}\,\mathbf{D}) = \frac{\langle\rho\rangle}{2}\frac{d}{dt}\left(\sum_{\alpha=1}^{3}\lambda_\alpha^2\right) + \langle\rho\rangle\,\mathtt{tr}\left(\sum_{\alpha=1}^{3}\lambda_\alpha^2\mu_\alpha\right) = 0.$$

System (6) is a conservative and Galilean invariant system of equations which is the counterpart of the Euler equations for the turbulent compressible flows. The equations for the Reynolds stress tensor (6_3) are rewritten in a simpler form admitting a clear physical interpretation.

4 An approximate model

In this Section, we derive a useful approximation of model (6) describing compressible turbulent flows for motions characterized by a small average vorticity.

4.1 Preliminaries

Equation (5) can be rewritten as :

$$\frac{d\mathbf{P}}{dt} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T \mathbf{P} + \mathbf{P}\,\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = [\; i\,(\mathrm{rot}\mathbf{U})\;\;,\;\mathbf{P}\,] \tag{20}$$

with

$$[\; i\,(\mathrm{rot}\mathbf{U})\;\;,\;\mathbf{P}\,] = \mathbf{P}\,i\,(\mathrm{rot}\mathbf{U}) - i\,(\mathrm{rot}\mathbf{U})\,\mathbf{P}, \quad i\,(\mathrm{rot}\mathbf{U}) = \frac{\partial \mathbf{U}}{\partial \mathbf{x}} - \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T.$$

Let τ be a characteristic time scale and ω a characteristic value of the mean vorticity norm $\|\mathrm{rot}\mathbf{U}\|$; we assume that

$$\tau\,\omega \ll 1. \tag{21}$$

Relation (21) is verified, in particular, for motions which are close to one-dimensional ones. Equation (20) gets the form

$$\frac{d\mathbf{P}}{dt} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T \mathbf{P} + \mathbf{P}\,\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = 0. \tag{22}$$

Using the solution (7) when $\mathbf{P}_0(\mathbf{X}) = \mathbf{I}$, we consider the Reynolds stress tensor in the form

$$\mathbf{P} = \sum_\alpha \nabla\;\varphi_\alpha \otimes \nabla\;\varphi_\alpha,$$

where index α ranges over a finite number of integers and φ_α are generalized Lagrangian coordinates :

$$\frac{d\varphi_\alpha}{dt} = 0.$$

Covectors

$$\mathbf{b}_\alpha^T = \frac{\partial \varphi_\alpha}{\partial \mathbf{x}}$$

verify the identity

$$\frac{d\mathbf{b}_\alpha^T}{dt}+\mathbf{b}_\alpha^T\frac{\partial \mathbf{U}}{\partial \mathbf{x}}=\mathbf{0},$$

corresponding to a zero Lie derivative of $\mathbf{b}_\alpha^T$ with respect to the mass average velocity. In such a case, Eq. (22) is identically verified. Symmetric tensor $\mathbf{P}$ is determined by six scalar fields $\varphi_\alpha,\ \alpha\in\{1,\cdots,6\}$. If, initially, vectors $\nabla\ \varphi_\alpha,\ \alpha\in\{1,2,3\}$ constitute an orthogonal system corresponding to the eigenvectors of $\mathbf{P}$, we can choose $\varphi_\alpha=0$, $\alpha\in\{4,5,6\}$ and consequently,

$$\mathbf{P}=\sum_{\alpha=1}^{3}\nabla\ \varphi_\alpha\otimes\nabla\ \varphi_\alpha. \tag{23}$$

It is worth to note that since the Reynolds stress tensor is positive definite, it can be considered as a metric tensor of a Riemannian space associated with the metric

$$P_{ij}=\sum_{\alpha=1}^{3}\frac{\partial\varphi_\alpha}{\partial x_i}\frac{\partial\varphi_\alpha}{\partial x_j}.$$

This metric is flat because

$$\sum_{i,j}P_{ij}dx^i dx^j=\sum_{\alpha=1}^{3}(d\varphi_\alpha)^2$$

It is interesting to note that this special structure of the Reynolds stress tensor implies a variational structure of equations (6).

4.2 The Hamilton principle

The aim of this Section is to prove that in special form (23), System (6) admits a variational formulation which is similar to the one of capillary fluids [3, 4, 5, 6, 9, 17, 20]. However, in our case, the expression of the *capillary energy* is determined by the gradients of three scalar order parameters transported along the trajectories of the mean flow.
We consider the specific internal energy in the form

$$e_i=e_i\left(\langle\rho\rangle\right).$$

The mean density is submitted to the constraint

$$\frac{\partial\left\langle\rho\right\rangle}{\partial t}+\mathrm{div}\left(\left\langle\rho\right\rangle\mathbf{U}\right)=0.$$

Let us define the specific turbulent energy as

$$e_T=\sum_{\alpha=1}^{3}\frac{\left|\nabla\varphi_\alpha\right|^2}{2},$$

where scalars φ_α are submitted to the constraint

$$\frac{d\varphi_\alpha}{dt} = 0, \qquad \alpha \in 1, 2, 3.$$

For a material volume D_t of the mean motion, the Hamilton action calculated between times t_1, t_2 is

$$a = \int_{t_1}^{t_2} \left(\iiint_{D_t} \langle\rho\rangle \mathcal{L} \; dv \right) dt,$$

with the specific Lagrangian

$$\mathcal{L} = \left(\frac{1}{2} \mathbf{U}^T \mathbf{U} - e_i - e_T \right).$$

The fluid motion is a C^2- diffeomorphisme ϕ from a three-dimensional space D_0 into the physical space D_t :

$$\mathbf{x} = \phi\left(\mathbf{X}, t\right) \quad \text{or} \quad x_i = \phi_i(X_1, X_2, X_3, t)\,, \quad i \in \{1, 2, 3\}.$$

Let a one-parameter family of virtual motions denoted by $\{\phi_\varepsilon\}$, possessing continuous derivatives up to the second order and expressed in the form :

$$\mathbf{x} = \mathbf{\Phi}\left(\mathbf{X}, t; \varepsilon\right),$$

with $\varepsilon \in \mathcal{O}$, where $\mathcal{O}$ is an open interval containing 0 and such that $\mathbf{\Phi}\left(\mathbf{X}, t; 0\right) = \phi\left(\mathbf{X},t\right)$ (the real motion of the continuous medium is obtained when $\varepsilon = 0$). The derivation with respect to ε when $\varepsilon = 0$ is denoted by δ. Derivation δ is named variation and the virtual displacement $\delta\phi$ is the variation of the motion of the medium. At time t, the virtual displacement of the particle $\mathbf{x}$ is $\delta\mathbf{x}$ obtained when $\delta\mathbf{X} = 0$ and $\delta\varepsilon = 1$ at $\varepsilon = 0$; the virtual displacement corresponds to the field of tangent vectors to D_t

$$\mathbf{x} \in D_t \to \zeta = \psi(\mathbf{x}) \equiv \frac{\partial \mathbf{\Phi}}{\partial \varepsilon}\,|_{\varepsilon=0} \in T_{\mathbf{x}}(D_t)\,,$$

where $T_{\mathbf{x}}(D_t)$ is the tangent vector bundle to D_t at $\mathbf{x}$.
The Hamilton principle reads : *for each vector field of virtual displacements such that ζ and its derivatives vanish at the boundary $\partial\Omega$ of Ω,*

$$\delta a = \; 0.$$

We have the following general results (see [1, 4, 11, 16]) :

$$\begin{cases} \delta \left(\displaystyle\int_{t_1}^{t_2} \left(\iiint_{D_t} \langle\rho\rangle\, \mathcal{L} \; dv \right) dt \right) = \displaystyle\int_{\Omega} \langle\rho\rangle\, \delta\mathcal{L} \; dv \, dt, \\[2ex] \delta \mathbf{U} = \dfrac{d\zeta}{dt}, \\[2ex] \delta \left\langle\rho\right\rangle = -\left\langle\rho\right\rangle \mathrm{div}\zeta, \\[2ex] \delta \left(\dfrac{\partial\varphi_\alpha}{\partial\mathbf{x}} \right) = -\dfrac{\partial\varphi_\alpha}{\partial\mathbf{x}} \dfrac{\partial\zeta}{\partial\mathbf{x}}, \end{cases}$$

where $\Omega = [t_1, t_2] \times D_t$ and $\int_\Omega$ is the quadruple integral in the time-space domain Ω. Consequently,

$$\langle\rho\rangle\, \delta\mathcal{L} = \langle\rho\rangle\, \mathbf{U}^T \frac{d\zeta}{dt} + \langle\rho\rangle^2 \frac{\partial e_i}{\partial\langle\rho\rangle}\, \mathrm{div}\zeta + \sum_{\alpha=1}^{3} \langle\rho\rangle\, \frac{\partial\varphi_\alpha}{\partial\mathbf{x}} \frac{\partial\zeta}{\partial\mathbf{x}} \left(\frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^{\mathrm{T}}.$$

Let us denote $\langle p\rangle = \langle\rho\rangle^2\, \partial e_i/\partial\langle\rho\rangle$, the mean pressure scalar field of the fluid; due to the identities,

$$\begin{cases}
\langle\rho\rangle\, \mathbf{U}^T \dfrac{d\zeta}{dt} \equiv \dfrac{\partial\left(\langle\rho\rangle\, \mathbf{U}^T\zeta\right)}{\partial t} + \mathrm{div}\left(\langle\rho\rangle\, \mathbf{U}^{\mathrm{T}}\zeta\right) - \langle\rho\rangle \dfrac{\mathrm{d}\mathbf{U}^{\mathrm{T}}}{\mathrm{dt}}\, \zeta, \\[2ex]
\langle p\rangle\, \mathrm{div}\zeta \equiv \mathrm{div}\left(\langle \mathrm{p}\rangle\ \zeta\right) - \dfrac{\partial\,\langle \mathrm{p}\rangle}{\partial\mathbf{x}}\, \zeta, \\[2ex]
\displaystyle\sum_{\alpha=1}^{3} \langle\rho\rangle\, \frac{\partial\varphi_\alpha}{\partial\mathbf{x}} \frac{\partial\zeta}{\partial\mathbf{x}} \left(\frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^T \equiv \sum_{\alpha=1}^{3} \mathrm{tr}\left(\langle\rho\rangle\, \left(\frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^T \frac{\partial\varphi_\alpha}{\partial\mathbf{x}} \frac{\partial\zeta}{\partial\mathbf{x}}\right), \\
\displaystyle\equiv \sum_{\alpha=1}^{3} \mathrm{div}\left(\langle\rho\rangle\, \left(\frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^T \frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\, \zeta\right) - \sum_{\alpha=1}^{3} \mathrm{div}\left(\langle\rho\rangle\, \left(\frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^T \frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)\zeta\,.
\end{cases}$$

Stokes' formula implies that terms $\dfrac{\partial\left(\langle\rho\rangle\, \mathbf{U}^T\zeta\right)}{\partial t}$, $\mathrm{div}\left(\langle\rho\rangle\, \mathbf{U}^{\mathrm{T}}\zeta\right)$, $\mathrm{div}\left(\langle \mathrm{p}\rangle\ \zeta\right)$ and $\mathrm{div}\left(\langle\rho\rangle\, \left(\dfrac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^T \dfrac{\partial\varphi_\alpha}{\partial\mathbf{x}}\, \zeta\right)$ can be integrated on $\partial\Omega$ where ζ vanishes. We get for each field of virtual motion,

$$\delta a = -\int_\Omega \left\{\langle\rho\rangle \frac{d\mathbf{U}^T}{dt} + \frac{\partial\,\langle p\rangle}{\partial\mathbf{x}} + \sum_{\alpha=1}^{3} \mathrm{div}\left(\langle\rho\rangle\, \left(\frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)^T \frac{\partial\varphi_\alpha}{\partial\mathbf{x}}\right)\right\}\zeta\ dv\, dt,$$

and the fundamental lemma of variational calculus yields the equation of motion

$$\langle\rho\rangle\, \frac{d\mathbf{U}}{dt} + \nabla\,\langle p\rangle + \mathrm{div}\left(\langle\rho\rangle \sum_{\alpha=1}^{3} \nabla\varphi_\alpha \otimes \nabla\varphi_\alpha\right) = 0;$$

the variational formulation of the approximate system is established.

5 Conclusion

The equations of fluid turbulent motions take three equations into account: the equation of mass balance (1_1), the balance equation of average momentum (1_2), and the Reynolds stress tensor equation of evolution (1_3); this last equation has been the object of our study. If the turbulent sources are neglected, the turbulent fluid motion is a superposition of the mean motion and turbulent fluctuations. The eigenvectors of the Reynolds stress tensor carry the fluctuations associated

with the mean flow deformation. The amplitudes of turbulent deformations are defined by the eigenvalues of the Reynolds stress tensor. The equations for the directions of turbulent fluctuations are reminiscent of a gyroscopic type equation for the motion of a free rigid body (Equation (19_1)). The amplitude evolution of turbulent deformations are determined by the diagonal values μ_α of the mean rate of deformation tensor $\mathbf{D}$ expressed in the eigenvector basis of the Reynolds stress tensor. The turbulence increases with the time when $\mu_\alpha < 0$, and decreases when $\mu_\alpha > 0$. In the particular case of incompressible fluid motions we have $\mathtt{tr}\,\mathbf{D} = 0$, and hence there always exists a direction in which the turbulence is increasing while in other directions the turbulence is decreasing. Over the past two decades great progress has been made in understanding many aspects of the kinematics and dynamics of a wide variety of turbulent flows as a result of access to the mean velocity gradient tensor (see [10, 19]). Such an access could be used for an experimental determining these eigenvalues.
In the case of a small mean vorticity, a new approximate model admitting a variational formulation is derived.

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